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Non-growing Preferential Attachment Random Graphs

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Abstract

We consider an edge rewiring process which is widely used to model the dynamics of scale-free weblike networks. This process uses preferential attachment and operates on sparse multigraphs with n vertices and m edges. We prove that its mixing time is optimal and develop a framework which simplifies the calculation of graph properties in the steady state. The applicability of this framework is demonstrated by calculating the degree distribution, the number of self-loops and the threshold for the appearance of the giant component.

1 Introduction

The study of complex networks has attracted considerable attention during the last 10 years. Recently there are signals suggesting that a new phase in the network development, importance and proliferation is on the way. Some networks like the Internet and the WWW became mission critical. Even some military operations are partially supported through the Internet not to mention news reports and information from geographical regions under crisis. The importance of complex networks has led to a deeper research showing that the behavior and the growth of such networks are far from being purely random. It often follows certain topological and structural patterns discovered in the theory of scale-free networks and small worlds [2, 19].

During the past decade an interplay between statistical analysis of real world networks and a formal study of stochastic models has turned out to be promising (see [5] for a survey on methods used for the analysis of such models). The models are designed to simulate the dynamic change in the network and to produce the properties observed in real world networks. A formal analysis is important in order to verify the quality of the model by comparing it with empirically collected data [15]. Unfortunately, the well understood random graph models $G_{n,p}$ and $G_{n,m}$ do neither produce the features that are observed in real world networks nor are they designed to simulate the dynamics of those networks. The research community is therefore challenged to design and analyze better models.

The most discussed property of complex networks is the scale-free degree distribution [2]. The first model that creates graphs with a scale-free degree distribution was the Barabási-Albert model which was rigorously analyzed in [4]. The Barabási-Albert model adds in every time step a vertex and connects it to the existing graph by choosing a vertex with preferential attachment. This is believed to model the growing phase of a network quite well. However, it is known [18] that several complex networks reach a state of saturation in which they do not grow anymore but still evolve. Unfortunately there are almost no rigorous results known for models of non-growing networks. In this paper we make a first step towards analyzing such a model.

A widely accepted stochastic process [6, 11, 9] which uses preferential attachment and considers multigraphs with a stable amount of vertices and edges is defined as follows. At every time step one of the m edges in the multigraph is selected uniformly at random and one of its half edges is rewired to a new vertex, which is selected at random from n vertices with probability proportional to a preference function $f(d(v)) : \{1, ..., 2m\} \to \mathbb{R}$, where $d(v)$ is the degree of vertex v. If there is more than one graph we write $d_G(v)$ for the degree of vertex v in graph G. The symbol $\langle f \rangle$ stands for the average value $(\sum_v f(d(v)))/n$ of f. This process is called SESP (Simple Edge Selection Process) and has the following formal definition.

Process 1.1 (SESP)**.** *The following steps are repeated at each time unit:*
1. An edge e_i is selected uniformly at random.
2. An end vertex v_i of e_i is selected uniformly at random.
3. A vertex v_l is selected at random with probability $\frac{f(d(v_l))}{n\langle f \rangle}$.
4. The edge e_i is rewired from v_i to v_l.

In [9] Evans and Plato solved a mean field equation for the degree distribution of Process 1.1. We complement the understanding of this model with a rigorous study of the graph properties in the steady state of the process. We believe that SESP represents one of the simplest possible processes that is nevertheless principally different than classical random graphs. This claim is backed by the fact that if the preference function is set to a constant then every multigraph appears with equal probability in the stationary distribution which is equivalent to a multigraph version of $G_{n,m}$. The next natural step, which is tackled in this paper, is to consider linear preference functions.

1.1 SESP Markov chain

The parameters of Process 1.1 are the number of vertices n, the number of edges m, the preference function f and the input graph G_0. To reformulate the model as a Markov chain, we define the state space Ω that contains all multigraphs with n labeled vertices and $2m$ labeled half edges. We split each edge e in the middle and denote its two half edges as e^1 and e^2. Let the vertices and half edges be labeled $v_1, ..., v_n$ respectively $e_1^1, e_1^2, ..., e_m^1, e_m^2$. Then a labeled multigraph $x \in \Omega$ is defined as $(H(v_1), ..., H(v_n))$ where $H(v_i)$ denotes the set of half edges at vertex v_i. The transition probabilities between two states in Ω explicitly follow from Process 1.1. The result of step 1. and 2. in Process 1.1 is that every half edge is selected with probability $1/(2m)$. Therefore we will use the following equivalent definition of the process for the remainder of this paper.

Process 1.2 (SESP Markov Chain)**.** *The following steps are repeated at each time unit:*
1. A half edge e_i^j (where $1 \leq i \leq m$ and $j \in \{0, 1\}$) is selected uniformly at random from all half edges.
2. A vertex v_l is selected at random with probability $\frac{f(d(v_l))}{n\langle f \rangle}$.
3. The half edge e_i^j is rewired to v_l.

Note that for $x, y \in \Omega$ there is a non-zero transition probability if and only if there exist at most one half edge e_i^1 that is at a different vertex (v_p) in x than in y (v_q). Then a transition from x to y happens only if e_i^1 is selected in step 1 and v_q is selected in step 2 of Process 1.2. Thus,

$$P(x, y) = \frac{1}{2m} \frac{f(d_x(v_q))}{n\langle f \rangle} \text{ and } P(y, x) = \frac{1}{2m} \frac{f(d_y(v_p))}{n\langle f \rangle} \tag{1}$$

where $d_s(v)$ denotes the degree of vertex v in state s. The Markov chain is irreducible if and only if $f(k) > 0$ for all $k \in \{0, ..., 2m\}$ because then every state can be reached from every other state in at

most $2m$ transitions. It is furthermore aperiodic (and therefore ergodic) because a transition from state $G \in \Omega$ back into itself has positive probability for all states. It follows from the fundamental Markov chain theorem (see for example Theorem 4.16 on page 58 of [13]) that the chain will converge into its unique stationary distribution π.

All our results in this article are for linear preference functions $f(k) = ak + b$ where $a \geq 0$ and $b > 0$ are constants such that $a + b = 1$. In [9] Evans et al. explained that selection of a vertex of degree k with probability $p_p \frac{k}{2m} + p_r \frac{1}{n}$, where p_p and p_r are probabilities which add up to 1, is the most general form of linear preferential attachment. Because of

$$\frac{f(k)}{n\langle f\rangle} = \frac{ak+b}{n(a\overline{k}+b)} = \frac{a}{a+\frac{b}{\overline{k}}}\frac{k}{2m} + \frac{b}{a\overline{k}+b}\frac{1}{n} = p_p\frac{k}{2m} + p_r\frac{1}{n}$$

for

$$a := \frac{p_p}{\overline{k}p_r + p_p} \text{ and } b := \frac{\overline{k}p_r}{\overline{k}p_r + p_p}$$

our choice of the preference function is equivalent. We require that b is strictly larger than zero because otherwise the Markov process is not irreducible and will converge to a particular set of states. When $b = 0$ the analysis of the process is simple as the graph condensates to a state where all edges are connected to one single vertex. Complex networks are usually sparse, we therefore assume that the average degree $\overline{k} = \frac{2m}{n}$ is constant.

In this paper we study graph properties in the stationary distribution of the SESP Markov chain and can therefore neglect the initial graph G_0. We justify this proceeding by showing that the process reaches the stationary distribution fast. Note that the authors of [9] chose a different approach. They derived the dynamic of the degree distribution over time by solving a meanfield equation for the number of vertices of degree k at time t.

Let $G_{n,a,\overline{k}}$ denote the random graph model in which multigraphs on n vertices and $m = \frac{n\overline{k}}{2}$ edges are distributed accordingly to the stationary distribution of the SESP Markov chain with preference function $f(k) = ak + b$ (where $b := 1 - a$ and $0 \leq a < 1$). Note that for such a function the average is

$$\langle f\rangle = a\overline{k} + b = a(\overline{k} - 1) + 1 \ . \tag{2}$$

We use $G \sim G_{n,a,\overline{k}}$ to denote that graph G is drawn at random with distribution $G_{n,a,\overline{k}}$. For a graph property $\mathcal{P}$ we say that $G_{n,a,\overline{k}}$ has $\mathcal{P}$ asymptotically almost surely (a.a.s.) or with high probability (whp) whenever the probability that $G \sim G_{n,a,\overline{k}}$ has $\mathcal{P}$ is $1 - o(1)$ where n is the relevant parameter for all asymptotic notations unless otherwise stated. The natural question when examine a graph property is: "For what parameters a and $\overline{k}$ does $G_{n,a,\overline{k}}$ exhibit the property as n tends to infinity?"

1.2 Main Results

Our first result shows that the Markov chain converges fast which legitimates our approach of looking only at the stationary distribution.

Theorem 1. *The mixing time $t_{mix}(\epsilon)$ of SESP is bounded by*

$$t_{mix}(\epsilon) \leq \log(n\overline{k}\epsilon^{-1})nc_1$$

where $c_1 := \frac{\overline{k}(a(\overline{k}-1)+1)}{1-a}$.

This upper bound on the mixing time is asymptotically optimal as every half edge needs to be rewired at least once to have a non zero probability that two processes with initial graphs of

distance $2m$ are in the same state. It is well known (see for example the coupon collector problem in chapter 2.2 of [13]) that $O(m \log m)$ steps are needed to have a high probability on the event that all $2m$ half edges have been selected at least once.

Inequalities that give a measure for the concentration of a quantity around its expectation appeared helpful in the study of random graph models. A 1-Lipschitz function $f : \Omega \to \mathbb{R}$ has the property that for all $x, y \in \Omega$

$$\frac{|f(x) - f(y)|}{h(x, y)} \leq 1$$

where $h(x, y)$ is the shortest path metric in the Markov chain. We observe that many important graph measures like the number of isolated vertices, the chromatic number or the number of self-loops are Lipschitz functions. In this work we develop the following concentration inequality for 1-Lipschitz functions.

Theorem 2. *For all 1-Lipschitz functions $f : \Omega \to \mathbb{R}$ and all $u > 0$ it holds that if $G \sim G_{n,a,\overline{k}}$ then*

$$\Pr[|f(G) - \mathbb{E}[f(G)]| \geq u] \leq 2e^{-\frac{u^2}{nc_1}}$$

where $c_1 := \frac{\overline{k}(a(\overline{k}-1)+1)}{1-a}$.

The degree distribution is an important topic in complex network research. In this text we use the random variable $N(k)$ to denote the number of vertices of degree k in $G_{n,a,\overline{k}}$. In the following theorem we derive the expected number of vertices of a given degree in $G \sim G_{n,a,\overline{k}}$.

Theorem 3. *For every $\varepsilon > 0$, $G \sim G_{n,a,\overline{k}}$ has a.a.s. no vertex of degree larger than $k_{max} := \left(\frac{2+\varepsilon}{\log c_2}\right) \log m$ and for all $0 \leq k \leq k_{max}$ the expected number of vertices of degree k has the following asymptotic form*

$$\mathbb{E}[N(k)] = \Theta\left(n \cdot \frac{1}{\left(1 + \frac{b}{a\overline{k}}\right)^k} \cdot k^{-1+\frac{b}{a}}\right) \quad .$$

One consequence of Theorem 2 is that the degree distribution is concentrated around its expectation.

Corollary 4. *For $0 \leq k \leq 2m$ and $G \sim G_{n,a,\overline{k}}$*

$$\Pr[|N(k) - \mathbb{E}[N(k)]| \geq \sqrt{n \log n}] \leq 2n^{-\frac{1}{4c_1}} \tag{3}$$

where $c_1 := \frac{\overline{k}(a(\overline{k}-1)+1)}{1-a}$.

The configuration model for random graphs is the state of the art method for the examination of random graphs with a specific degree distribution. For a degree sequence $d_1, \ldots, d_n$ where d_i denotes the degree of v_i we write $G \sim Conf(d_1, \ldots, d_n)$ if G is chosen at random out of all multigraphs with probability defined by the configuration model. Theorem 5 states that in $G_{n,a,\overline{k}}$ all multigraphs of a given degree sequence have the same probability as in the configuration model.

Theorem 5. *Let $G_{SESP} \sim G_{n,a,\overline{k}}$ and for an arbitrary degree sequence $d_1, \ldots, d_n$ such that $\sum d_i = n\overline{k}$ let $G_{Conf} \sim Conf(d_1, \ldots, d_n)$. Then for all multigraphs G with degree sequence $d_1, \ldots, d_n$*

$$\Pr[G_{SESP} = G | d_{G_{SESP}}(v_1) = d_1, \ldots, d_{G_{SESP}}(v_n) = d_n] = \Pr[G_{Conf} = G] \quad .$$

In [16, 17] Molloy and Reed derived the threshold for the appearance of the giant component in the configuration model for a large class of random graphs. In Section 3.2 we apply the framework provided by the previous theorems to show that $G_{n,a,\overline{k}}$ is in that class. This results in a precise formulation of the threshold as function of a and $\overline{k}$ in Lemma 11. We believe that there exists a simple formulation of the threshold, namely that there is a giant component if $\overline{k} > (1-a)$ and that all components are small if $\overline{k} < (1-a)$ as formulated in Conjecture 12. We prove the conjecture for two important special cases of the preference function.

In Section 2 we prove Theorem 1 and Theorem 2. In Section 3 we derive Theorem 3 and Corollary 4. Then in Section 3.1 we prove Theorem 5 by looking at symmetries in the transition diagram of the Markov chain. All these results are used to derive the number of self-loops and the threshold for the appearance of the giant component in Section 4. At the end we compare our results to previous work and draw a conclusion.

2 Mixing time of SESP

We use the following standard consequence (see for example Theorem 2.1 in [8]) of the Coupling Lemma [1] to prove an upper bound on the convergence time of the SESP Markov chain.

Lemma 6. *Let (X_t, Y_t) be a coupling of a Markov chain and let ρ be any integer valued metric defined on $\Omega \times \Omega$. Suppose that there exist $\beta \leq 1$ such that $\mathbb{E}[\rho(X_{t+1}, Y_{t+1})] \leq \beta\rho(X_t, Y_t)$ for all t and all $(X_t, Y_t) \in \Omega \times \Omega$. Let D be the maximum value that ρ achieves on $\Omega \times \Omega$. If $\beta < 1$ then the mixing time $t_{mix}(\epsilon)$ of the Markov chain satisfies*

$$t_{mix}(\epsilon) \leq \frac{\log(D\epsilon^{-1})}{(1-\beta)} \ .$$

We now use the above tool to prove the optimal mixing time of the SESP Markov chain.

Proof of Theorem 1. The transition probabilities (1) imply that $P(x,y) > 0$ if and only if $P(y,x) > 0$ and the Markov chain therefore has an underlying undirected graph induced on the states Ω where x and y are connected by an edge if and only if $P(x,y) > 0$. Let $h(x,y)$ denote the length of the shortest path between the two states x and y in this undirected graph. The central idea of the proof is to define a coupling $(X, X') \in \Omega \times \Omega$ of the Markov chain such that we can use the shortest path distance $h(x,x')$ (which is clearly a metric) and apply Lemma 6. Let X be an exact copy of the SESP Markov chain with the transition probabilities described in (1). Our goal is to design X' in such a way that the expected distance between X and X' decreases after every single transition. Note that for all $x, x' \in \Omega$ the distance $h(x,x')$ counts exactly the number of half edges that are in x connected to different vertices than in x'. We are therefore tempted to define X' such that it selects the same half edge as X and rewires it to the same vertex as in X. But this would not be a valid coupling as the marginal transition probabilities of X' would be different than in the original SESP Markov chain. We therefore have to change this idea slightly to create a formally correct coupling of the Markov chain.

To describe the transition probabilities of X' we denote $d(v)$ as the degree of vertex v in X and $d'(v)$ as the degree of v in X'. Assume without loss of generality that in X we selected half edge e_i^1 and rewired it to vertex v_p. Then if $f(d(v_p)) \geq f(d'(v_p))$ we rewire in X' e_i^1 to vertex v_p with probability

$$\frac{f(d'(v_p))}{f(d(v_p))} \text{ and with probability } \left(1 - \frac{f(d'(v_p))}{f(d(v_p))}\right) \frac{f(d'(v_q)) - f(d(v_q))}{\Delta}$$

$$\text{to } v_q \in U' := \{v | f(d'(v)) > f(d(v))\}$$
$$\text{where } \Delta := \sum_{v_l \in U'} f(d'(v_l)) - f(d(v_l)) \ .$$

We furthermore define

$$U := \{v | f(d(v)) > f(d'(v))\}$$

and observe that

$$\Delta := \sum_{v_l \in U'} f(d'(v_l)) - f(d(v_l)) = \sum_{v_l \in U} f(d(v_l)) - f(d'(v_l))$$

for linear preference functions. If $f(d(v_p)) < f(d'(v_p))$ we rewire e_i^1 to v_p.

It remains to show that the marginal transition probabilities of X' are equal to the transition probabilities of SESP. The marginal edge selection probability in X' is clearly $\frac{1}{2m}$ for every half edge. The vertex selection probability for a vertex v_i with $f(d'(v_i)) \leq f(d(v))$ is

$$\frac{f(d(v_i))}{n\langle f\rangle}\frac{f(d'(v_i))}{f(d(v_i))} = \frac{f(d'(v_i))}{n\langle f\rangle} \ .$$

For a vertex with $f(d'(v_i)) > f(d(v))$ the vertex selection probability is

$$\begin{aligned}
&\frac{f(d(v_i))}{n\langle f\rangle} + \sum_{v_l \in U} \frac{f(d(v_l))}{n\langle f\rangle}\left(1 - \frac{f(d'(v_l))}{f(d(v_l))}\right)\frac{f(d'(v_i)) - f(d(v_i))}{\Delta} \\
&= \frac{f(d(v_i))}{n\langle f\rangle} + \frac{f(d'(v_i)) - f(d(v_i))}{n\langle f\rangle\Delta}\sum_{v_l \in U} f(d(v_l)) - f(d'(v_l)) = \frac{f(d'(v_i))}{n\langle f\rangle} \ .
\end{aligned}$$

We observe that the distance between the states x and x' of the coupling (X, X') decreases if a half edge that is on a different vertex in X than in X' is selected and rewired to the same vertex in X and X'. Similarly the distance increases if a half edge which is on the same vertex in both states is selected and rewired to different vertices. Therefore we can calculate the expectation of $h(X_{t+1}, X'_{t+1})$ as a function of $h(X_t, X'_t)$

$$\begin{aligned}
&\mathbb{E}[h(X_{t+1}, X'_{t+1})] \\
&= h(X_t, X'_t) - \frac{h(X_t, X'_t)}{2m}\left(1 - \sum_{v_l \in U}\frac{f(d(v_l))}{n\langle f\rangle}\left(1 - \frac{f(d'(v_l))}{f(d(v_l))}\right)\right) \\
&+ \left(1 - \frac{h(X_t, X'_t)}{2m}\right)\sum_{v_l \in U}\frac{f(d(v_l))}{n\langle f\rangle}\left(1 - \frac{f(d'(v_l))}{f(d(v_l))}\right) \\
&= h(X_t, X'_t) - \frac{h(X_t, X'_t)}{2m} + \frac{1}{n\langle f\rangle}\sum_{v_l \in U} f(d(v_l)) - f(d'(v_l)) \\
&= h(X_t, X'_t) - \frac{h(X_t, X'_t)}{2m} + \frac{a}{n\langle f\rangle}\sum_{v_l \in U} d(v_l) - d'(v_l) \\
&\leq h(X_t, X'_t) - \frac{h(X_t, X'_t)}{2m} + \frac{a \cdot h(X_t, X'_t)}{n\langle f\rangle} \\
&= h(X_t, X'_t)\left(1 - \frac{(a2m + nb) - a2m}{2mn\langle f\rangle}\right) = h(X_t, X'_t)\left(1 - \frac{b}{n\overline{k}\langle f\rangle}\right) \ . \qquad (4)
\end{aligned}$$

There are $2m$ half edges, which can be connected to different vertices in x and y, which means that the distance between two states is always at most $D := 2m$. We can therefore finish the proof by applying Lemma 6 and (2) to the coupling (X, X') which satisfies (4). □

The following Lemma was proven as Corollary 4.4 in [14].

Lemma 7. *Suppose that there is a constant $0 < \beta < 1$ and two copies (X, X') of a Markov chain with Wasserstein distance*

$$dist_W(X_1, X_1') \leq \alpha$$

for all $x_0, x_0' \in \Omega$ with $h(x_0, x_0') = 1$. Then if the Markov chain has stationary distribution π it holds for all $u > 0$ and every 1-Lipschitz function f that

$$\Pr(|f(X) - \mathbb{E}[f(X)]| \geq u) \leq 2e^{-u^2(1-\beta^2)/2\beta^2}$$

where X is a random variable with distribution π.

Proof of Theorem 2. It follows from the Kantorovich-Rubinstein theorem (see [14] or for a more general discussion [7]) that the Wasserstein distance at time 1 of two SESP Markov chains Y_t and Y_t' with starting state y_0 and y_0' is bounded by

$$dist_W(Y_1, Y_1') \leq \mathbb{E}[h(X_1, X_1')|x_0 = y_0, x_0' = y_0']$$

where (X, X') is an arbitrary coupling of the Markov chain. Hence we can use the coupling defined above and conclude that for x_0, x_0' with $h(x_0, x_0') = 1$ the Wasserstein distance is bounded by

$$dist_W(X_1, X_1') \leq \left(1 - \frac{b}{n\overline{k}\langle f\rangle}\right) \ .$$

Considering Lemma 7 this proves the theorem. □

Note that many graph properties like the number of isolated vertices, the number of self-loops or the maximum degree are 1-Lipschitz functions. For some properties we derive the same bound by applying Theorem 5 and Azuma's inequality. However, this result is only helpful for functions f where $\mathbb{E}[f] = \Omega\left(\sqrt{n \log n}\right)$ because otherwise we get already by Markov's inequality a stronger bound.

3 The Degree Sequence

In order to analyze the degree sequence in the stationary distribution we calculate the probability $P(k)$ that an arbitrary vertex is of degree k. Then we show that the number of vertices of degree k is sufficiently concentrated around its expectation $\mathbb{E}[N(k)] = n \cdot P(k)$. In one step of Process 1.2 a particular vertex v gains a half edge with probability $f(d(v))/(n\langle f\rangle)$ and looses a half edge with probability $d(v)/(2m)$. For a linear preference function the average value $\langle f\rangle = a\overline{k} + b$ is a constant that does not depend on the degree distribution $P(k)$. We therefore observe that the degree of any vertex v at time $t+1$ depends only on the degree of v at time t. Thus, the random variable that measures the degree of a vertex v at time t is an ergodic Markov chain which has the unique stationary distribution $P(k)$ for $0 \leq k \leq 2m$. We will refer to this chain as the Degree Markov chain (DMC). The states of the DMC are the possible degrees that the vertex can take ($\Omega_{DMC} := \{0, 1, ..., 2m\}$). Note that SESP consist of n instances of the DMC which are all controlled by one single source of randomness.

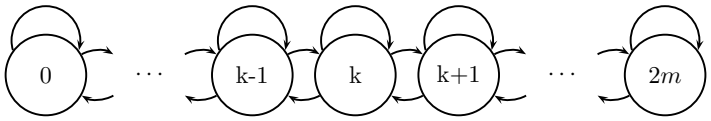

Figure 1: Transition diagram of the Degree Markov Chain.

The probability for a transition from state $x = k$ to state $y = k + 1$ is

$$P(x, y) = \left(1 - \frac{k}{2m}\right) \frac{f(k)}{n\langle f\rangle}$$

as this transition requires the event that no half edge of the vertex is removed and the event that the vertex is selected for rewiring. The remaining transition probabilities result from analogical reasoning. Table 3 lists all transition probabilities of the DMC. We now use the DMC to prove

Table 1: Transition probabilities for the Degree Markov Chain (DMC).

x	y	$P(x,y)$	$P(y,x)$
k	k+1	$\left(1 - \frac{k}{2m}\right) \frac{f(k)}{n\langle f\rangle}$	$\frac{k+1}{2m}\left(1 - \frac{f(k+1)}{n\langle f\rangle}\right)$
k	k-1	$\frac{k}{2m}\left(1 - \frac{f(k)}{n\langle f\rangle}\right)$	$\left(1 - \frac{k-1}{2m}\right) \frac{f(k-1)}{n\langle f\rangle}$
k	k	$\left(1 - \frac{k}{2m}\right)\left(1 - \frac{f(k)}{n\langle f\rangle}\right) + \frac{k}{2m}\frac{f(k)}{n\langle f\rangle}$	

Theorem 3.

Proof of Theorem 3. The DMC is a Birth-and-Death chain (see [13] Section 2.5) and therefore reversible. Hence we can calculate $P(k)$ from $P(0)$ by

$$\begin{aligned} P(k) =& \frac{P(k-1,k)}{P(k,k-1)} P(k-1) = \left(\prod_{i=0}^{k-1} \frac{P(i,i+1)}{P(i+1,i)}\right) \cdot P(0) \\ =& \frac{\prod_{i=0}^{k-1}(2m-i)f(i)}{\prod_{i=1}^{k} i(n\langle f\rangle - f(i))} P(0) \ . \end{aligned}$$

The normalization condition gives us

$$P(0) = \frac{1}{\sum_{k=0}^{2m} \prod_{i=1}^{k} \frac{f(i-1)\cdot(2m-i+1)}{i\cdot(n\langle f\rangle - f(i))}} \ . \tag{5}$$

Hence for $0 \le k \le 2m$

$$P(k) = \frac{\prod_{i=1}^{k} \frac{f(i-1)\cdot(2m-i+1)}{i\cdot(n\langle f\rangle - f(i))}}{\sum_{j=0}^{2m} \prod_{i=1}^{j} \frac{f(i-1)\cdot(2m-i+1)}{i\cdot(n\langle f\rangle - f(i))}} \ . \tag{6}$$

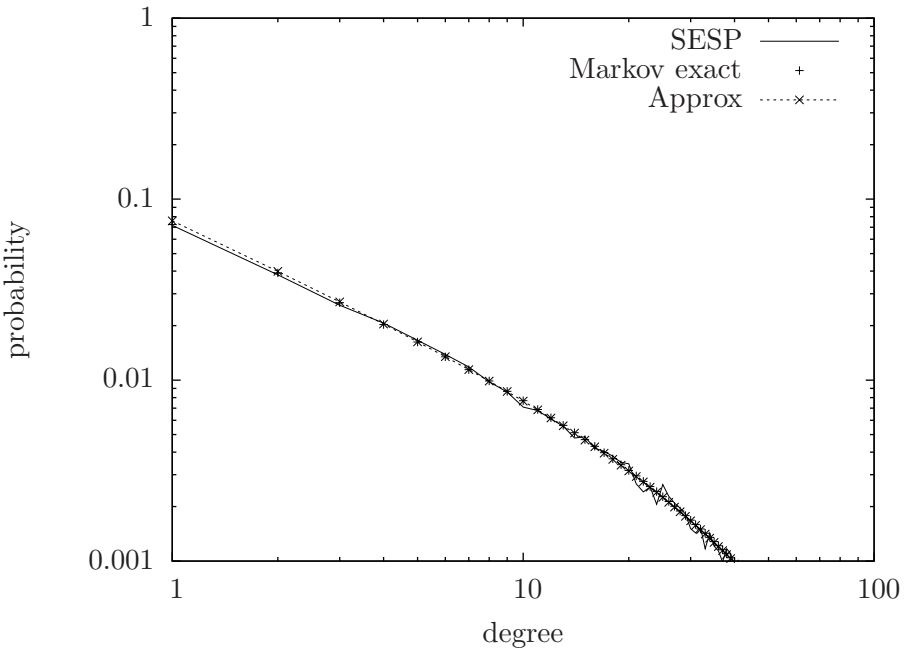

Figure 2: (color online) In this illustration we see the exact stationary distribution of the Markov chain (6) against the approximation (10) and the degree distribution of a network after 10^7 simulation steps of SESP. All data is for the preference function $f(k) = 0.9k + 0.1$ and a network of size $n = 1000$ and $m = 2000$.

For $a = 0$ we observe by straight forward calculation that $G_{n,a,\overline{k}}$ is the binomial random graph for which

$$P(k) = \binom{2m}{k} \left(\frac{1}{n}\right)^k \left(\frac{n-1}{n}\right)^{2m-k} . \tag{7}$$

For $a > 0$ we can rewrite the numerator of (6) using the Gamma function

$$\begin{aligned}
\prod_{i=1}^{k} \frac{f(i-1)\cdot(2m-i+1)}{i\cdot(n\langle f\rangle - f(i))} &= \frac{\prod_{i=0}^{k-1}(ai+b)}{k!} \frac{\prod_{i=2m-k+1}^{2m} i}{\prod_{i=2m-k}^{2m-1}(ai+b(n-1))} \\
&= \frac{a^k \prod_{i=0}^{k-1}\left(i+\frac{b}{a}\right)}{k!} \frac{\Gamma(2m+1)}{\Gamma(2m-k+1)} \frac{1}{a^k \prod_{i=2m-k}^{2m-1}(i+\frac{b}{a}(n-1))} \\
&= \frac{\Gamma(k+\frac{b}{a})}{\Gamma(\frac{b}{a})\Gamma(k+1)} \frac{\Gamma(2m+1)}{\Gamma(2m-k+1)} \frac{\Gamma(2m-k+\frac{b}{a}(n-1))}{\Gamma(2m+\frac{b}{a}(n-1))} .
\end{aligned}$$

By applying Stirling's approximation of the Gamma function (see Lemma 13 in Section A) we can show that

$$\frac{\Gamma(k+\frac{b}{a})}{\Gamma(\frac{b}{a})\Gamma(k+1)} \frac{\Gamma(2m+1)}{\Gamma(2m-k+1)} \frac{\Gamma(2m-k+\frac{b}{a}(n-1))}{\Gamma(2m+\frac{b}{a}(n-1))} = O\left(c_2^{-k}\cdot k^{-1+\frac{b}{a}}\right) \tag{8}$$

where $c_2 := 1 + \frac{b}{a\overline{k}}$. Note that Lemma 13 states that for $0 \le k \le \left(\frac{2+\varepsilon}{\log c_2}\right)\log m$ the asymptotic upper bound is tight. We use (8) to simplify the denominator of (6). There is some constant c such

that

$$\sum_{j=0}^{2m}\prod_{i=1}^{j}\frac{f(i-1)\cdot(2m-i+1)}{i\cdot(n\langle f\rangle-f(i))}=\sum_{k=0}^{2m}O\left(\left(1+\frac{b}{a\overline{k}}\right)^{-k}\right)\leq c\cdot\sum_{k=0}^{\infty}\left(1+\frac{b}{a\overline{k}}\right)^{-k}\ .$$

As this infinite sum converges, we conclude that

$$P(k)=O\left(c_2^{-k}\cdot k^{-1+\frac{b}{a}}\right)\ . \tag{9}$$

The exponential cut-off (c_2^{-k}) of the degree distribution already indicates that the probability for vertices of large degree is small. We now show that for $\varepsilon>0$, there are a.a.s. no vertices of degree larger than $k_{max}:=\left(\frac{2+\varepsilon}{\log c_2}\right)\log m$. By union bound and (9) the probability that there is a vertex of degree at least k_{max} is at most

$$n\sum_{i=k_{max}}^{2m}P(k_{max})\leq\frac{2m\cdot n}{\left(1+\frac{b}{a\overline{k}}\right)^{k_{max}}}=o(1)\ .$$

Therefore in the stationary distribution of SESP the maximum degree of a graph is asymptotically almost surely at most $\left(\frac{2+\varepsilon}{\log c_2}\right)\log m$ and $P(k)$ simplifies for all $k\leq\left(\frac{2+\varepsilon}{\log c_2}\right)\log m$ to

$$P(k)=\Theta\left(c_2^{-k}\cdot k^{-1+\frac{b}{a}}\right)\ . \tag{10}$$

□

The number of vertices of degree k divided by two is a function from the state space of the Markov chain to the real numbers. The function is 1-Lipschitz because in one step of SESP the number of vertices of degree k increases or decreases by at most two and therefore

$$\forall_{x,x'\in\Omega}:\frac{|N_x(k)-N_{x'}(k)|}{2d(x,x')}\leq 1\ .$$

Corollary 4 is a simple consequence of the above observation and Theorem 2. Note that we use $N_x(k)$ for the number of vertices of degree k in graph x which is slightly different than our usual notation.

3.1 Relation to the Configuration Model

The configuration model (see [3, 12]) is the standard method for studying random regular graphs and random graphs of a given degree sequence. For a degree sequence $d(v_1), d(v_2), ..., d(v_n)$ the first step is to construct the n vertices containing the $2m$ labeled half edges. Then a configuration (a pairing of the $2m$ half edges) is chosen uniformly at random from all possible configurations. The multigraph that corresponds to the chosen configuration can be constructed by inserting an edge between every pair of half edges. There are

$$(2m-1)!!=(2m-1)\cdot(2m-3)\cdot\dots\cdot 3\cdot 1=\frac{(2m)!}{2^m m!} \tag{11}$$

different configurations for $2m$ half edges.

Recall that the state space of the SESP Markov chain Ω consists of all possible multigraphs on n labeled vertices and $2m$ labeled half edges. A multigraph $s\in\Omega$ is represented by a vector

which contains for every vertex v the set of adjacent half edges $H(v)$ as defined in Section 1.1. We partition these states such that two graphs are in the same partition if and only if they exhibit the same degree sequence $d(v_1), \ldots, d(v_n)$. Let Ω' be the state space defined by all degree sequences and for a partitioning Ω' of the state space Ω let $\phi(v) : \Omega \to \Omega'$ be the unique x in Ω' such that v is an element of x. We will use the following Lemma to prove that in the stationary distribution of SESP all multigraphs of the same degree sequence appear with equal probability as in the configuration model. This means that we can use the configuration model to study $G_{n,a,\overline{k}}$. Note that a degree distribution is often given as the number of vertices of a certain degree $N(1), N(2), \ldots, N(2m)$ and not as the sequence of the degrees $d(v_1), \ldots, d(v_n)$. But obviously for a fixed $N(1), N(2), \ldots, N(2m)$ every sequence of degrees (which matches $N(1), N(2), \ldots, N(2m)$) appears with the same probability as it is only a matter of relabeling the initial graph.

Lemma 8. *If Ω' is a partition of the ergodic Markov chain Ω, P, π which satisfies*

$$\forall_{x,y\in\Omega'}\forall_{u,v\in y} : \sum_{s\in x} P(s,u) = \sum_{s\in x} P(s,v) \tag{12}$$

and

$$\forall_{x,y\in\Omega'}\forall_{u,v\in x} : \sum_{s\in y} P(u,s) = \sum_{s\in y} P(v,s) \tag{13}$$

then

$$\forall_{v\in\Omega} : \pi(v) = \frac{\sum_{u\in\phi(v)} \pi(u)}{|\phi(v)|} \quad .$$

Proof. Because of ergodicity it is sufficient to show that for all $v \in \Omega$

$$\sum_{w\in\Omega} P(w,v) \frac{\sum_{u\in\phi(w)} \pi(u)}{|\phi(w)|} = \frac{\sum_{s\in\phi(v)} \pi(s)}{|\phi(v)|}$$

is satisfied. First note that

$$\forall_{x\in\Omega'}\forall_{u\in x}\forall_{v\in\Omega} : \sum_{w\in x} P(w,v) \overset{(12)}{=} \frac{1}{|\phi(v)|} \sum_{s\in\phi(v)} \sum_{w\in x} P(w,s) \overset{(13)}{=} \frac{|x|}{|\phi(v)|} \sum_{s\in\phi(v)} P(u,s) \quad . \tag{14}$$

Hence, for arbitrary $v \in \Omega$

$$\begin{aligned}
&\sum_{w\in\Omega} P(w,v) \frac{\sum_{u\in\phi(w)} \pi(u)}{|\phi(w)|} = \sum_{x\in\Omega'} \frac{1}{|x|} \sum_{u\in x} \pi(u) \sum_{w\in x} P(w,v) \\
&\overset{(14)}{=} \frac{1}{|\phi(v)|} \sum_{x\in\Omega'} \sum_{u\in x} \pi(u) \sum_{s\in\phi(v)} P(u,s) \\
&= \frac{1}{|\phi(v)|} \sum_{s\in\phi(v)} \sum_{w\in\Omega} P(u,s)\pi(u) = \frac{\sum_{s\in\phi(v)} \pi(s)}{|\phi(v)|}
\end{aligned}$$

always holds. □

We now use Lemma 8 to prove that for every degree sequence $s \in \Omega'$ and every multigraph with labeled vertices and half-edges $G_i \in s$ the stationary distribution is $\pi(G_i) = \frac{\sum_{u\in s} \pi(u)}{|s|}$.

Proof of Theorem 5. To apply Lemma 8 we only need to show that the SESP Markov chain satisfies (12) and (13). We first proof (12) for arbitrary degree sequences $x, y \in \Omega'$.

We first assume that $x \neq y$. If there is a positive transition probability from a state s in x to a state w in y this means that there is one half edge e and two vertices v and v' such that

$$H_s(v) = H_w(v) \cup \{e\} \text{ and } H_w(v') = H_s(v') \cup \{e\}$$

and for all other vertices $u \in V - \{v, v'\}$ $H_s(u) = H_w(u)$. Then a transition from s to w happens if and only if e is selected and rewired to v'. Thus, there are non zero transition probabilities from x to y if and only if there is exactly one pair of vertices v, v' such that

$$d_x(v) = d_y(v) + 1 \text{ and } d_x(v') = d_y(v') - 1 \ .$$

Then each multigraph $w \in y$ has exactly $d_y(v')$ different multigraphs $s \in x$ which have transition probability $P(s, w) = \frac{1}{2m} \frac{a \cdot d_x(v') + b}{n(a\overline{k} + b)}$ and hence (12) is satisfied.

If $x = y$ then for all $s, w \in x$ $P(s, w) > 0$ implies that $s = w$ and therefore (12) is satisfied because $P(s, s) = P(w, w) = \sum_{v \in V} \frac{d_x(v)}{2m} \frac{a \cdot d_x(v) + b}{n(a\overline{k} + b)}$.

The above observations can be applied analogically to show that (13) is also satisfied.

We observe that for a given degree sequence $d(v_1), d(v_2), \ldots, d(v_n)$ every labeling of the half edges appears with the same probability. Choosing a labeling of the half edges uniformly at random and connecting the two half edges of each edge is equivalent to choosing a pairing of the half edges uniformly at random out of all $(2m - 1)!!$ pairings. □

3.2 Properties of the Degree Sequence

In [16, 17] Molloy and Reed defined the following natural properties for degree sequences of large graphs. An **asymptotic degree sequence** is a sequence of integer valued functions $\mathbb{D} = d_0(n), d_1(n), ...$ such that

1. $d_i(n) = 0$ for $i \geq n$
2. $\sum_{i \geq 0} d_i(n) = n$.

An asymptotic degree sequence $\mathbb{D}$ is **well-behaved** if:

1. $\mathbb{D}$ is smooth, which means that there exist constants λ_i such that

$$\lim_{n \to \infty} \frac{d_i(n)}{n} = \lambda_i \ ,$$

   where n denotes the total number of vertices and $d_i(n)$ the number of vertices of degree i.

2. For all $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$ and for all $i \geq 0$:

$$\left| \frac{i(i-2)d_i(n)}{n} - i(i-2)\lambda_i \right| < \varepsilon \ .$$

3.

$$L(\mathbb{D}) = \lim_{n \to \infty} \sum_{i \geq 1} \frac{i(i-2)d_i(n)}{n}$$

exists and the sum approaches the limit uniformly i.e. if $L(\mathbb{D})$ is finite then for all $\varepsilon > 0$, there exists i^*, n_0 such that for all $n > n_0$

$$\left| \sum_{i=1}^{i^*} \frac{i(i-2)d_i(n)}{n} - L(\mathbb{D}) \right| < \varepsilon$$

or if $L(\mathbb{D})$ is infinite then for all $T > 0$, there exists i^*, n_0 such that for all $n > n_0$

$$\sum_{i=1}^{i^*} \frac{i(i-2)d_i(n)}{n} > T \ .$$

An asymptotic degree sequence $\mathbb{D}$ is **sparse** if

$$\sum_{i \geq 0} i d_i(n)/n = K + o(1)$$

for some constant K.

Lemma 9. *For constant $\overline{k}$ the degree sequence of $G_{n,a,\overline{k}}$ is whp a well-behaved and sparse asymptotic degree sequence.*

Proof. By Theorem 5 we can fix a graph $G \sim G_{n,a,\overline{k}}$ by first exposing its degree sequence $N(0), \ldots, N(2m)$ and then choosing a random multigraph of that degree sequence by using the configuration model. We show that the degree sequence is well-behaved whenever the maximum degree is at most $k_{max} := \left(\frac{3}{\log c_2}\right) \log m$ and

$$\forall_{0 \leq k \leq k_{max}} : n \cdot P(k) - \sqrt{n \log n} \leq N(k) \leq n \cdot P(k) + \sqrt{n \log n} \tag{15}$$

holds. Note that $n \cdot P(k) = \mathbb{E}[N(k)]$ and the two events are therefore by Corollary 4 and Theorem 3 satisfied with probability at least

$$1 - \left(\frac{3}{\log c_2}\right) \log m \cdot 2n^{-\frac{1}{4c_1}} - o(1) = 1 - o(1) \ .$$

It remains to show that the degree sequence $d_0(n), \ldots, d_{2m}(n) = N(0), \ldots, N(2m)$ is well-behaved if (15) is satisfied and the maximum degree is at most k_{max}.

Clearly $\sum_{i \geq 0} d_i(n) = n$ and $d_i(n) = 0$ for $i > n-1$. Let $\lambda_i := P(i)$ for $i \leq k_{max} := \left(\frac{3}{\log c_2}\right) \log m$ and $\lambda_i = 0$ for larger i. Because of (15)

$$\frac{d_i(n)}{n} = \lambda_i \pm O\left(\frac{\sqrt{n \log n}}{n}\right) \text{ and therefore } \lim_{n \to \infty} \frac{d_i(n)}{n} = \lambda_i \ .$$

For $i > k_{max}$ we have that

$$\left| \frac{i(i-2)d_i(n)}{n} - i(i-2)\lambda_i \right| = 0$$

while if $i \leq k_{max}$ then

$$\left| \frac{i(i-2)d_i(n)}{n} - i(i-2)\lambda_i \right| = \left| i(i-2)\frac{O(\sqrt{n \log n})}{n} \right| \overset{i \leq k_{max}}{=} o(1)$$

and therefore the second condition for a well behaved degree sequence is satisfied. For the third condition we inspect $L(\mathbb{D})$.

$$L(\mathbb{D}) = \lim_{n\to\infty} \left(\sum_{i\geq 1} i(i-2)\lambda_i + \sum_{i=1}^{k_{max}} i(i-2)\frac{O(\sqrt{n\log n}}{n}) \right)$$
$$= \lim_{n\to\infty} \left(\sum_{i\geq 1} i(i-2)\lambda_i + o(1) \right) = \sum_{i=1}^{\infty} i(i-2)\lambda_i \ . \tag{16}$$

This means that the limit exists and because $\lambda_i = O\left(\left(1+\frac{b}{a\overline{k}}\right)^{-i}\right)$ the sum

$$\sum_{i\geq 1}^{\infty} i(i-2)\lambda_i$$

converges. Hence for all $\epsilon > 0$ there exist a constant i^* such that

$$\sum_{i=i^*+1}^{\infty} i(i-2)\lambda_i < \epsilon$$

and therefore for some n_0 and all $n > n_0$

$$\left| \sum_{i=1}^{i^*} \frac{i(i-2)d_i(n)}{n} - L(\mathbb{D}) \right| \overset{(16)}{=} \left| \sum_{i=1}^{i^*} i(i-2)\frac{d_i(n)}{n} - \sum_{i=1}^{\infty} i(i-2)\lambda_i \right|$$
$$= \left| \sum_{i=1}^{i^*} i(i-2)\left(d_i(n)/n - \lambda_i\right) - \sum_{i=i^*+1}^{\infty} i(i-2)\lambda_i \right|$$
$$= \left| o(1) - \sum_{i=i^*+1}^{n} i(i-2)\lambda_i \right| < \epsilon \ .$$

□

Note that Fernholz and Ramachandran [10] defined a similar set of properties for which they proved that the diameter of random graphs satisfying those properties is $c\ln n + o(\ln n)$. By combining their results and our framework it might be possible to prove that the diameter of $G_{n,a,\overline{k}}$ is $O(\ln n)$.

4 Applications

In this section we demonstrate how our main results can be used to study properties of $G_{n,a,\overline{k}}$. We can use the configuration model (Theorem 5) and the degree distribution (Theorem 3 and Corollary 4) to calculate the expectation of random variables in $G_{n,a,\overline{k}}$. As long as those random variables can be formulated as 1-Lipschitz functions we can then apply Theorem 2 to derive bounds on their concentration. The weakness of SESP is that it is not restricted to simple graphs which means that it can produce graphs with self-loops. In [6], the authors indicated that the number of self-loops is small. In Section 4.1 we give a short rigorous proof of this fact. In Section 4.2 we use a result that was derived for the configuration model to study the emergence of the giant component in $G_{n,a,\overline{k}}$. This exemplifies how known results can be transferred from the configuration model to $G_{n,a,\overline{k}}$.

4.1 Self-loops

We now use the configuration model to derive the number of self-loops in $G_{n,a,\overline{k}}$. Note that in the configuration model the probability that two specific half-edges are connected is exactly

$$\frac{(2m-3)!!}{(2m-1)!!} = \frac{1}{2m-1} \ . \tag{17}$$

The expected number of self-loops attached to a vertex of degree k is $\binom{k}{2}\frac{1}{2m-1}$ as there are $\binom{k}{2}$ pairs of half edges which are connected with probability $\frac{1}{2m-1}$. We can calculate the expected number of self-loops on an arbitrary vertex by summing over all possible degrees k

$$\sum_k p(k)\binom{k}{2}\frac{1}{2m-1} = \frac{\left\langle\binom{k}{2}\right\rangle}{2m-1} \ .$$

Therefore, the expected number of self-loops in the graph is $\frac{n\cdot\left\langle\binom{k}{2}\right\rangle}{2m-1}$. In $G_{n,a,\overline{k}}$

$$\left\langle\binom{k}{2}\right\rangle = O\left(\sum_k^{2m} k^{1+\frac{b}{a}}\left(1+\frac{b}{a\overline{k}}\right)^{-k}\right) = O(1)$$

because the infinite sum

$$\sum_k^{\infty} k^{1+\frac{b}{a}}\left(1+\frac{b}{a\overline{k}}\right)^{-k}$$

converges.

4.2 Giant Component

In 1995 Molloy and Reed [16, 17] discovered the critical point for random graphs of a given asymptotic degree sequence. For a sequence of non-negative real numbers $\lambda_0, \lambda_1, \ldots$ they showed that a random graph that has approximately $\lambda_i n$ vertices of degree i has a.a.s. a giant component if $\sum i(i-2)\lambda_i > 0$ while if $\sum i(i-2)\lambda_i < 0$ a.a.s. all components are small.

Molloy and Reed introduced two lemmas on configurations to prove their results on random simple graphs. We merge those two lemmas (Lemma 3 and Lemma 7 in [16]) into the following one.

Lemma 10. *For a well-behaved and sparse asymptotic degree sequence $\mathbb{D}$ let $Q(\mathbb{D}) := \sum_{i\geq 0} i(i-1)\lambda_i$, $\sum_{i\geq 0} id_i(n)/n = K + o(1)$, $v = -Q(\mathbb{D})/K$, $R = 150/v^2$ and let F be a random configuration with n vertices and degree sequence $\mathbb{D}$. If $Q(\mathbb{D}) < 0$ and if for some function $0 \leq \omega(n) \leq n^{1/8-\epsilon}$ F has no vertices of degree greater than $\omega(n)$, then F a.a.s. has no component with more than $\alpha = \lceil R\omega(n)^2 \log n\rceil$ vertices. If $Q(\mathbb{D}) > 0$ then there exist constants $\zeta_1, \zeta_2 > 0$ dependent on $\mathbb{D}$ such that a.a.s. F has a component with at least $\zeta_1 n$ vertices and $\zeta_2 n$ cycles.*

In Section 3.2 we showed that $G \sim G_{n,a,\overline{k}}$ a.a.s. meets the conditions of Lemma 10 with $\lambda_i := P(i)$ for $i \leq k_{max} := \left(\frac{3}{\log c_2}\right)\log m$ and $P(i) = 0$ else. Therefore a graph $G \sim G_{n,a,\overline{k}}$ has a.a.s. a giant component if $Q(\mathbb{D}) > 0$ while if $Q(\mathbb{D}) < 0$ almost surely all components have size $o(n)$. This is formulated in the following lemma.

Lemma 11. *Let $Q = \sum_{k=1}^{k_{max}} k(k-2)P(k)$ where*

$$P(k) = \frac{\prod_{i=1}^{k}\frac{f(i-1)\cdot(2m-i+1)}{i\cdot(n\langle f\rangle - f(i))}}{\sum_{j=0}^{2m}\prod_{i=1}^{j}\frac{f(i-1)\cdot(2m-i+1)}{i\cdot(n\langle f\rangle - f(i))}} \text{ and } k_{max} := \left(\frac{3}{\log c_2}\right)\log m \ .$$

Then for $Q > 0$, $G \sim G_{n,a,\overline{k}}$ has a.a.s. a component of $\Theta(n)$ vertices while for $Q < 0$ there is a.a.s. no component with more than $O(\log^2 n)$ vertices.

Lemma 11 characterizes for which parameters a and $\overline{k}$ the giant component in $G_{n,a,\overline{k}}$ emerges. The goal is to determine a simpler function of a and $\overline{k}$ that decides whether $G_{n,a,\overline{k}}$ has a giant component. We believe that $Q < 0$ if $\overline{k} < b$ and that $Q > 0$ if $\overline{k} > b$.

Conjecture 12. *Let $G \sim G_{n,a,\overline{k}}$ then for $\overline{k} > b$ G has a.a.s. a component of size $\Theta(n)$ and for $\overline{k} < b$ there is a.a.s. no component of size larger than $O(\log^2 n)$.*

The cases $a = b$ and $b = 1$ as well as numerical calculations confirm this conjecture. We now show that the conjecture is true for $a = b$ and $b = 1$.

In (7) (Section 3) we derived that the degree distribution for $b = 1$ and $a = 0$ is

$$P(k) = \binom{2m}{k} \left(\frac{1}{n}\right)^k \left(\frac{n-1}{n}\right)^{2m-k} .$$

Let c be a constant. Then for $m = c(n-1)$ it follows that

$$\begin{aligned} Q &= \left(\frac{n-1}{n}\right)^{2c(n-1)} \sum_{i=1}^{k_{max}} i(i-2)\binom{2c(n-1)}{i}(n-1)^{-i} \\ &= \left(\frac{n-1}{n}\right)^{2c(n-1)} \left(-2c + \sum_{i=3}^{k_{max}} i(i-2)\binom{2c(n-1)}{i}(n-1)^{-i}\right). \end{aligned}$$

For $c < 1/2$ this means that $Q < 0$ because

$$-2c + \sum_{i=3}^{k_{max}} i(i-2)\binom{2c(n-1)}{i}(n-1)^{-i} \leq -2c + \sum_{i\geq 3} i(i-2)\frac{(2c)^i}{i!} < 0$$

while $Q > 0$ for $c > 1/2$ as

$$-2c + \sum_{i=3}^{k_{max}} i(i-2)\binom{2c(n-1)}{i}(n-1)^{-i} = -2c + \sum_{i=3}^{k_{max}} i(i-2)\frac{(2c)^i}{i!}(1-o(1)) > 0$$

for n large enough.

Another case for which we derive a threshold for the giant component is if $a = b = 1/2$. Then Q in Lemma 11 simplifies to

$$\begin{aligned} Q &= \frac{1}{c}\sum_{i=1}^{k_{max}} i(i-2)\prod_{k=1}^{i}\frac{(a(k-1)+b)(2m-k+1)}{k(a(2m-k)+b(n-1))} \\ &= \frac{1}{c}\frac{b2m}{a(2m-1)+b(n-1)}\left(-1 + \sum_{i=3}^{k_{max}}\frac{(i-2)}{(i-1)!}\prod_{k=2}^{i}\frac{\left(k-1+\frac{b}{a}\right)\left(\overline{k}-\frac{k-1}{n}\right)}{\overline{k}+\frac{b(n-1)}{an}-\frac{k}{n}}\right) . \end{aligned}$$

where c is the normalization term of $P(k)$. Therefore $Q \overset{(<)}{>} 0$ if and only if

$$Q' := \sum_{i=3}^{k_{max}} i(i-2)\prod_{k=2}^{i}\frac{\overline{k}-\frac{k-1}{n}}{\overline{k}+1-\frac{k+1}{n}} \overset{(<)}{>} 1 .$$

For $\overline{k} = 1/2 - \varepsilon$

$$Q' = \sum_{i=3}^{k_{max}} i(i-2) \prod_{k=2}^{i} \left(1 - \frac{1 - \frac{2}{n}}{\frac{3}{2} - \varepsilon - \frac{k+1}{n}}\right) < \sum_{i \geq 3} i(i-2) \left(\frac{1}{3}\right)^{i-1} = 1$$

because

$$f(z) := 1 - \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} = \sum_{i \geq 0} (i+2) i z^{i+1}$$

and $f(1/3) = 1$. For $\overline{k} = 1/2 + \varepsilon$

$$Q' = \sum_{i=3}^{k_{max}} i(i-2) \prod_{k=2}^{i} \left(1 - \frac{1 - \frac{2}{n}}{\frac{3}{2} + \varepsilon - \frac{k+1}{n}}\right) \overset{i+1 < \varepsilon n/2}{>} \sum_{i=3}^{k_{max}} i(i-2) \left(\frac{1+\varepsilon}{3+\varepsilon}\right)^{i-1} > 1$$

for n large enough as $f(\frac{1+\varepsilon}{3+\varepsilon}) = 1 + \varepsilon \left(\frac{3+\varepsilon}{2}\right)^2$.

5 Conclusion

In this paper we described a new way of looking at non-growing networks. By considering the stationary distribution of SESP, we defined a new random graph model $G_{n,a,\overline{k}}$ and developed a framework to study its properties. We underlined the relevance of this new approach by showing that the model has optimal mixing time and therefore reaches the stationary phase in $O(m \log m)$ steps. This means that a real world network (for which the model fits) is with high probability already in the stationary distribution.

A different approach was chosen in [9] where the authors solved a mean-field equation for the degree distribution after step t of the process. If we look at their solution for $t = \infty$ we get

$$P(k) \approx k^{-1 + \frac{p_r}{p_p}\overline{k}} \cdot p_p^k \tag{18}$$

for their preference function $f(k) = p_p \frac{k}{2m} + p_r \frac{1}{n}$ where $p_p + p_r = 1$. Note that for

$$p_p = \frac{a}{a + \frac{b}{\overline{k}}} \text{ and } p_r = \frac{b}{a\overline{k} + b}$$

the two models are equivalent. By substituting our p_p and p_r into (18)

$$P(k) \approx k^{-1 + \frac{\frac{b}{a\overline{k}+b}}{\frac{a}{a+\frac{b}{\overline{k}}}}\overline{k}} \cdot (\frac{a}{a + \frac{b}{\overline{k}}})^k = \frac{1}{\left(1 + \frac{b}{a\overline{k}}\right)^k} \cdot k^{-1+\frac{b}{a}}$$

we see that our solution confirms their result in the limit.

We believe that our framework can be used in similar manners as the methods known from traditional random graph models like the Erdős-Rényi model or the configuration model. The discussion of the giant component in Section 4.2 illustrates the building blocks of the framework. The equivalence to the configuration model (Theorem 5) and the concentration of 1-Lipschitz (Theorem 2) allows us to apply a result of [16]. We therefore used the degree distribution (derived in Section 3) to analyze the value of $\sum k(k-2)P(k)$.

One of the natural questions for future research would be how to develop an analogical theory for simple graphs. The natural attempt of ignoring step 4 of Process 1.1 whenever it would create a self-loop or a multiedge leads to considerable difficulties in the process analysis (see [11]). To surpass these obstacles, we see basically two possibilities. Either one studies multigraphs and uses the fact which we proved in the present paper that there is only very small number of self-loops (and analogically multiple edges) or one tries to study new meaningful processes which create scale-free distributions.

A Asymptotic Degree Sequence

The following technical lemma is used in Section 3.

Lemma 13. *For $n = 2m/\overline{k}$ and $0 \le k \le 2m$*

$$\frac{\Gamma(k+\frac{b}{a})}{\Gamma(k+1)}\frac{\Gamma(2m+1)}{\Gamma(2m-k+1)}\frac{\Gamma(2m-k+\frac{b}{a}(n-1))}{\Gamma(2m+\frac{b}{a}(n-1))} = O\left(k^{-1+\frac{b}{a}}c_2^{-k}\right)$$

and for $0 \le k \le \left(\frac{2+\varepsilon}{\log c_2}\right)\log m$

$$\frac{\Gamma(k+\frac{b}{a})}{\Gamma(k+1)}\frac{\Gamma(2m+1)}{\Gamma(2m-k+1)}\frac{\Gamma(2m-k+\frac{b}{a}(n-1))}{\Gamma(2m+\frac{b}{a}(n-1))} = \Theta\left(k^{-1+\frac{b}{a}}c_2^{-k}\right)$$

where $c_2 := 1 + \frac{b}{a\overline{k}}$.

Proof. We use Stirling's approximation

$$\Gamma(z) = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z\left((1+O\left(\frac{1}{z}\right)\right)$$

to obtain

$$\frac{\Gamma(k+\frac{b}{a})}{\Gamma(k+1)} = \sqrt{\frac{k+\frac{b}{a}}{k+1}}e^{1-\frac{b}{a}}\left(\frac{k+\frac{b}{a}}{k+1}\right)^k\frac{(k+\frac{b}{a})^{\frac{b}{a}}}{(k+1)}\Theta(1) = \Theta\left(k^{-1+\frac{b}{a}}\right) \tag{19}$$

where we used $\left(1+\frac{\frac{b}{a}-1}{k+1}\right)^k = \Theta(1)$ which follows from the known features of the exponential

function. For $n = 2m/\overline{k}$ we apply again the above Stirling's approximation to the remaining terms.

$$\begin{aligned}
&\frac{\Gamma(2m+1)}{\Gamma(2m-k+1)}\frac{\Gamma(2m-k+\frac{b}{a}(n-1))}{\Gamma(2m+\frac{b}{a}(n-1))} \\
=&\sqrt{\frac{(2m+1)\left(2m-k+\frac{b}{a}(n-1)\right)}{(2m-k+1)\left(2m+\frac{b}{a}(n-1)\right)}}\left(\frac{2m+1}{2m-k+1}\right)^{2m-k+1}(2m+1)^k \\
&\cdot\left(\frac{2m-k+\frac{b}{a}(n-1)}{2m+\frac{b}{a}(n-1)}\right)^{2m-k+\frac{b}{a}(n-1)}\frac{1}{\left(2m+\frac{b}{a}(n-1)\right)^k}\cdot\Theta(1) \\
=&\left(\frac{2m+1}{2m-k+1}\right)^{2m-k+1+\frac{1}{2}}(2m+1)^k\left(\frac{2m-k+\frac{b}{a}(n-1)}{2m+\frac{b}{a}(n-1)}\right)^{2m-k+\frac{b}{a}(n-1)+\frac{1}{2}} \\
&\cdot\frac{1}{\left(2m+\frac{b}{a}(n-1)\right)^k}\Theta(1) \\
=&\Theta\left(\left(\frac{2m+1}{2m-k+1}\right)^{2m-k+1+\frac{1}{2}}\left(\frac{2mc_2-k}{2mc_2}\right)^{2mc_2-k+\frac{1}{2}}c_2^{-k}\right)
\end{aligned}$$

where $c_2 := 1 + \frac{b}{a\overline{k}}$. We furthermore observe that

$$\phi(k) := \left(\frac{2m+1}{2m-k+1}\right)^{2m-k+1+\frac{1}{2}}\left(\frac{2mc_2-k}{2mc_2}\right)^{2mc_2-k+\frac{1}{2}}$$

is monotonically decreasing for $k \in \{0,\ldots,2m\}$ and m large enough. As $\phi(0) = 1$ we conclude that

$$\Theta\left(\left(\frac{2m+1}{2m-k+1}\right)^{2m-k+1+\frac{1}{2}}\left(\frac{2mc_2-k}{2mc_2}\right)^{2mc_2-k+\frac{1}{2}}c_2^{-k}\right) = O\left(c_2^{-k}\right) \ .$$

For $0 \le k \le \left(\frac{2+\varepsilon}{\log c_2}\right)\log m$

$$\begin{aligned}
\phi(k) =& \left(1-\frac{k}{2m+1}\right)^{-(2m+1)+k-1/2}\left(1-\frac{k}{2mc_2}\right)^{2mc_2-k+1/2} = (1+o(1))e^{-\frac{k^2-k/2}{2m+1}+\frac{k^2-k/2}{2mc_2}} \\
\ge&(1+o(1))e^{-\frac{k}{2m}\left(1-\frac{1}{c_2}\right)(k-1/2)} \ge (1+o(1))m^{-\frac{\left(\frac{2+\varepsilon}{\log c_2}\right)\left(1-\frac{1}{c_2}\right)(k-1/2)}{2m}} = \Theta(1) \ .
\end{aligned}$$

□

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