A STEPWISE TECHNIQUE FOR INVERSE PROBLEM IN OPTIMAL BOUNDARY CONTROL OF THERMAL SYSTEMS

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Abstract

In the paper a procedure for stepwise solution of an inverse heat conduction problem is proposed. Based on the procedure a control system for optimal boundary control of one class of thermal systems is submitted. The controlled thermal system is considered to be decomposed to two subsystems - a subsystem which is easy to control by a feedback using measurable outputs of the system and to a subsystem which distributed state - usually a spatial temperature distribution inside a heated material - is inacessible to direct measurement. The dynamics of the state is supposed to be described by a suitable distributed parameter model with a boundary excitation performed via the measurable system output. The control task then consists of optimal varying the measurable system output, that governs the boundary excitation of the distributed parameter subsystem, until it is calculated that the required shape of the distributed state has been reached. In the paper an optimal reference values for the system output, which should be tracked by a controller, are generated using stepwise technique for inversion of the distributed parameter model. The models of both subsystems are considered in continuous time, nonparametric convolutional integral forms. Using a spline approximation of the convolutional integral describing the first subsystem a predictive controller with receding horizon strategy is designed. The own inverse problem is solved by an iterative regularization method. The resulting stepwise procedure is illustrated on a one-dimensional problem of the boundary heating of a thin metal bar.

1 Introduction

A heating of solid materials is one of typical technological operations in industry. In many of this operations the aim is to remove the solid once a *centre* temperature of the solid has reached a specified value, or once the temperature within the solid has reached a specified spatial distribution. Moreover the heating process should be as quick as possible and optimal from technological, economical and ecological point of view. Modelling of this processes naturally leads to the problems of distributed parameter systems where state variables depend on spatial positions. More often than not only some of these spatially distributed variables are accesible to direct measurement. In this case the point is how to manipulate the unmeasured distributed state variables by utilizing other data measured on the thermal system only and the knowledge of the physical laws governing the process at hand.

The thermal state inside a solid object during the heating operation in a furnace is a typical example. Since it is difficult to monitor this state by routine measurement techniques, the control aim can be realized only by means of the surface temperature control. The thermal system can be easily decomposed to a pair of subsystems - a subsystem with measurable outputs (surface - boundary temperature of the heated object) which is easy to control by a feedback and a subsequent subsystem which is driven by the preceding subsystem and whose distributed state is inaccessible to direct measurement (inside temperature of heated object). Control of this inside temperature is achieved not by a feedback but by maintaining a pre-calculated temperature time profile at the boundary of the object.

The aim of this paper is to submit a predictive control system working with the measurable output of the thermal system and tracking a specified, optimally pre-calculated reference signal for boundary control of the system in order to obtain a required spatial temperature profile in the heated object at a selected time instants t_v. For given spatial temperature profile the reference signal for the system boundary control is obtained by inverting a distributed parameter model, which describes the dynamics of the unmeasurable temperature distribution in the second subsystem of the given thermal system. In the paper the inverse problem is converted to some *regularization* problem and is solved by a *stepwise* technique. This technique seems to be suitable for on-line control of thermal systems under a condition of stochastic disturbances acting on the controlled systems. The dynamics of the first subsystem is modelled by continuous-time convolutional integrals with finite-support kernels. The input and output signals of the subsystem are considered to be a polynomial splines. The B-splines are taken as base functions of these splines. The control synthesis is based on minimization of an integral continuous-time quadratic loss function, which after spline approximation is transformed to simple matrix quadratic form. To minimize the form the quadratic programming is employed. The allowed control input signal is then defined by a set of a suitable selected linear equality and inequality constraints which act on the vector of the polynomial coefficients of this signal.

In the following parts of this paper we will briefly discuss only the main ideas of the proposed boundary control system and for simplicity we will concentrate on simple one-dimensional heating problem: boundary heating of a thin metal bar. The heating apparatus is considered to be the controlled subsystem of the thermal system with the input signal u and the system output signal y - the measured boundary temperature of the metal bar. This temperature is the manipulated input to the second subsystem - heated metal bar - where the *unmeasured* spatial distribution of temperature in the direction of the bar length is modelled by known equation of the heat conduction.

2 Spline-based predictive controller

The controlled subsystem is assumed to be represented by the following time-invariant linear continuous convolution model:

$$\int_0^t a(t-\tau)\; y(\tau)\; d\tau + \\ + \int_0^t b(t-\tau)\; u(\tau)\; d\tau \;+\; o \;=\; \epsilon(t) \qquad (1)$$

where the finite-support kernels $a(t-\tau)$, $b(t-\tau)$ and the output signal $y(t)$ are considered to be approximated by suitable chosen spline functions, $\epsilon(t)$ is a noise of the model, o is an offset term. The input signal $u(t)$ is a polynomial spline of defined order which is generated by submitted control synthesis. The model (1) was originally developed in [3] mainly to improve discrete-time control with high sampling rate. The finite supports of the kernels a, b determine the finite lengths T_a, T_b of the past history of the signals y and u over which it is reasonable to perform integration in model (1) for any t.

Let's approximate the kernels and the signals in equation (1) by spline function. Then we can obtain the following discrete form of the original model:

$$c_a^T\; Q_a(t)\; c_y \;+\; c_b^T\; Q_b(t)\; c_u \;+\; o \;=\; e(t) \qquad (2)$$

where

c_a, c_b are vectors of model parameters (vectors of spline coefficients of the unknown kernels a, b)

c_u, c_y are vectors of spline coefficients of input and output signals u, y

$Q_a(t)$, $Q_b(t)$ are matrices of integrals of spline base functions products:

$$Q_a(t) \;=\; \int_{t-T_a}^{t} m_a(t-\tau)\; m_y^T(\tau)\; d\tau \qquad (3)$$

$$Q_b(t) \;=\; \int_{t-T_b}^{t} m_b(t-\tau)\; m_u^T(\tau)\; d\tau$$

$e(t)$ is noise term of the model modified by the approximation errors, the vectors m contain the spline base functions used for the approximation tasks. It is easy to show that the matrices $Q_a(t)$, $Q_b(t)$ do not depend on time and can be calculated in advance, before the regulation starts. Using this matrices we can form useful filters for the measured variables which enable us to keep low order models for identification and simultaneously high sampling rate for measurement and control.

To determine the optimal control input signal we will start with minimization of the following integral continuous-time quadratic loss function:

$$J(u) = \frac{1}{T_h} \int_{t_k}^{t_k+T_h} [(y(t) - y_r(t))^2 \ w_y(t) + \\ +(u(t) - u_r(t))^2 \ w_u(t)] \ dt, \ u(t) \ \in \ U_{ad}(t) \qquad (4)$$

where:
$y_r(t)$, $u_r(t)$ are reference signals; $w_y(t)$, $w_u(t)$ are weighting functions; T_h is a control horizon and $U_{ad}(t)$ is a set of allowed control input signals.

Let's consider the input and output signal in (4) (together with reference values) in the form of spline function. The projected control input signal will be wanted in its piecewise polynomial representation. Then after substitution to (4) we can find that:

$$J(p_h) = (y_h - y_h^r)^T \ Q_y \ (y_h - y_h^r) + \\ +(p_h - p_h^r)^T \ Q_u \ (p_h - p_h^r), p_u \ \in \ C_{ad}(T_h) \qquad (5)$$

where the penalty matrices Q_y, Q_u fully depend on the used spline base functions and can be calculated in advance; the vector p_h contains the coefficients of all polynomial pieces which form the projected control input signal on the time interval $[t_k, t_k + T_h]$ and the vector y_h includes the sampled values of the continuous-time output signal $y(t)$ to be predicted over the above time interval. To minimize the matrix form (5) the quadratic programming technique is well suited. Two classes of constraints are simultaneously used:
- the constraints which are inevitable to formulate the input signal as the spline function of given order
- the constraints which are due to physical limitations on the actuator or process (more frequently amplitude and rate limitations).

The quadratic programming technique can easily cover other types of constraints which are interesting for practice. Remarcable *tuning knob* in the spline based synthesis is a distance between the spline knots of the projected input signal in the time interval $[t_k, t_k + T_h]$:
- the polynomial pieces of the input signal can be projected on the same time interval as they will be really applied
- the polynomial pieces can be projected on time intervals which are a multiple of the interval as they will be really applied; it means that nonrealised (fake) control periods have been inserted into control horizon $[t_k, t_k + T_h]$.

The positive features of the technique consist in significant reduction of computation burden of quadratic programming and in zero control weighting for the control of non-minimum phase systems. For details see [6].

3 Formulation of the inverse problem

Let the behaviour of the unmeasured temperature field of the metal bar $s(x,t)$ at the time instant t and the position x of the bar is described by the parabolic partial differential equation:

$$\begin{aligned} &\frac{\partial}{\partial t} s(x,t) - a^2 \frac{\partial^2}{\partial x^2} s(x,t) + b s(x,t) = 0 \\ &s(x,t_0) = s_0(x), s(0,t) = y(t), \frac{\partial s}{\partial x}(L,t) = 0 \\ &0 \le x \le L, \quad t \ge t_0, \quad a \ne 0 \\ &a^2 = \frac{\lambda}{c.\rho}, \quad b = \frac{h}{c.\rho} \end{aligned} \qquad (6)$$

with known Green's function $G(x, \xi, t)$:

$$G(x,\xi,t) = \frac{2}{L} \sum_{n=0}^{\infty} \sin r_n x \sin r_n \xi \exp\left(-bt - a^2 {r_n}^2 t\right)$$

$$r_n = (2n+1)\frac{\pi}{2L}$$

where: L is the length of the bar; λ is thermal conductivity coefficient; c is specific heat; ρ is specific mass of the bar and h is heat-transfer coefficient.
Then the solution of the equation can be given in the following integral form:

$$s(x,t) = \int_{t_0}^{t} \int_0^L \ G(x, \xi, t-\tau) \ w(\xi, \tau) \ d\tau \qquad (7)$$

where $w(x,t)$ is a *standardizing* function (see [1]):

$$w(x,t) = s_0(x)\delta(t) - a^2 \delta'(x) y(t) \qquad (8)$$

which includes an exiting function, boundary and initial conditions and $\delta(.)$ is Dirac function. The heating of the bar is controlled through the boundary temperature $y(t) = s(0,t)$ and the task is to find such function $y(t)$ - boundary heating of the bar - which ensures us attainment of the required spatial distribution of the bar temperature $s(x,t)$ at specified time instant t_v. In this situation the relation (7) simplifies to the following form:

$$s(x, t_v) = \int_{t_0}^{t_v} \frac{\partial}{\partial \xi} \ G(x, \xi, t_v - \tau) \mid_{\xi=0} \ y(\tau) \ d\tau + \\ + \int_0^L \ G(x, \xi, t_v - t_0) \ s_0(\xi) \ d\xi \qquad (9)$$

where

$$s_0(x) = s(x, 0)$$

is given initial condition. The second part of equation (9) is known in the case of known initial condition. Let's denote it as $s_c(x,t_v)$ and define modified state $s_m(x,t_v)$ as:

$$s_m(x,t_v) = s(x,t_v) - s_c(x,t_v)$$

then

$$s_m(x,t_v) = \tag{10}$$
$$= -a^2 \int_{t_0}^{t_v} \frac{\partial}{\partial \xi} G(x,\xi,t_v-\tau) \mid_{\xi=0} y(\tau) d\tau$$

The last equation can be written in an operator form:

$$s_m = Ay \qquad s_m \in S \quad , \; y \in Y \subseteq Z \tag{11}$$

where A is the linear integral operator of relation (10), Z and S are Hilbert spaces, Y is closed convex set, build by a priori limitations of this task . The relation (11) represents an integral equation of the first type and solution of this equation fulfils the definition of the *ill-posed problems* in the Hadamard's sense. Therefore it is necessary to use some *regularization* method, which will give satisfactory results. In this paper we employ a method of Tikhonov [7], where the task of solving the equation (11) is replaced by the task of minimization of following *smoothing functional* $M_\alpha[y]$:

$$M_\alpha[y] = \| A_h y - s_{m\delta} \|^2 + \alpha \| y \|^2 \tag{12}$$

where $\alpha > 0$ is the regularization parameter.

A_h is an operator which approximates the operator A with defined error h, that means

$$\| A_h - A \| \leq h, \tag{13}$$

$s_{m\delta}$ is the left side of (11) which is specified by error δ:

$$\| s_m - s_{m\delta} \| \leq \delta, \tag{14}$$

Let's define so-called *generalized deviation* as :

$$\rho(\alpha) = \| A_h \; y_\alpha - s_{m\delta} \|^2 -$$
$$-(\delta + h \| y_\alpha \|^2) - (\mu \; (s_{m\delta}, A_h))^2 \tag{15}$$

where

$$\mu \; (s_{m\delta}, A_h) = \inf_{y \in Y} \; \| A_h y - s_{m\delta} \|$$

is the *degree of inconsistency*.

The regularization parameter α of the smoothing functional is chosen by *generalized principle of deviation*, which is following. If the condition:

$$\| s_{m\delta} \|^2 > \delta^2 + (\mu(s_{m\delta}, A_h))^2$$

is not fulfilled,the approximate solution of the equation (11) is $y = 0$. If the condition (3) is fulfilled, so the generalized deviation (3) has a positive root α^* and solution of the equation (11) is *minimum* y_{α^*} of the smoothing functional (12).

The solution of the equation (11) can be find by following iterative procedure :

1. Choose of arbitrary (sufficiently large) value of the parameter α
2. Minimization of the functional $M_\alpha[y]$ on the bounded region $Y \subseteq Z$
3. Evaluation of the generalized deviation $\rho(\alpha)$
4. Search for the root α^* of the equation $\rho(\alpha) = 0$ with accuracy ε, it means check the condition:

   $$\mid \; \rho(\alpha) \; \mid \leq \; \varepsilon \tag{16}$$

   where $\varepsilon = C * \delta$ and $C < 1$ is a constant which depends on the desired accuracy of the root α^*
5. If the condition (16) is not fulfilled the procedure is repeated from the point 2. Otherwise the *minimum* y_{α^*} is the solution of the equation (11).

As regards to the accuracy of the proposed procedure the fulfillment of the inequality (14) for found solution $y(t)$ is always guaranteed for suitably chosen operator error h and the inaccuracy δ.

4 Stepwise technique for the inverse problem

In preceding section the inverse task for the distributed model (6) was transformed to the problem of solving the operator equation (11) for a given left side. The solution $y_{\alpha^*} = y_r(t)$ of this equation forms the reference signal for optimal boundary control of the heated bar. The iterative regularization method used for solving equation (11) is valid only for in advance given and constant integral bounds t_0, t_v. This fact is necessary to take into account in designing the generator of the reference signal $y_r(t)$ for on-line boundary control of the thermal system. One way is to base the generator structure on *stepwise triggering* the inversion task in ekvidistantly located discrete time instants t_v. The distance between the time instants determines the integral bounds in the numerical solution of the equation (11) and in next explanation we will call the distance *inversion horizon* T_v. From practical point of view the length of the horizon T_v depends on several factors:

-technological needs for the heating process and the goal of the heating

-dynamical properties of the thermal system

-time behaviour of disturbances acting on the measurable system output.

In the process of the stepwise triggering of the inversion task with the time period T_v it is necessary to know at the particular starting time instant a *true* profile of the unmeasurable temperature distribution $s(x,t)$ in the heated bar. The true profile $s(x,t)$, which is really reached at the end of a preceding period, creates the initial condition for the inversion in a subsequent period. Because the temperature profile $s(x,t)$ is not measured its true time developement can be only *simulated* using a responce of the model (6) due to the really measured system output signal $y(t)$. For numerical calculation of the true responce $s(x,t)$ it is advantageous to utilize again the operator form (11) of the model. The length of the time interval during which the integration in (11) with the real signal $y(t)$ is performed we will call a *simulation horizon* T_m. For numerical reasons it is suitable to choose $T_m = T_v/ni$, where ni is given integer.

The starting point for the numerical solution of the simulation and the inversion tasks consists in suitable discretization of the basic relation (11) and its transformation to a matrix form. The resulting matrix form oriented to the simulation we will call a *simulation* model and the matrix form aimed at the inversion we will call an *inversion* model. Based on the above models the generator of the reference signal $y_r(t)$ is constructed. The required profiles $s_z(x)$ enter the generator with time period T_v and the real measured signal $y(t)$ enters the generator with period T_m.

5 Discretization of the simulation model

Consider the time instants t_j, $j = 1, 2, \dots$, in which the simulation tasks have to performed and let $\Delta t_j = t_j - t_{j-1} = T_m$.Then based on equation (10) for the time $\langle t_{j-1}, t_j\rangle$ it holds:

$$s(x,t_j) = s_m(x,t_j) + s_c(x,t_j) \tag{17}$$

$$\begin{aligned} s_m(x,t_j) &= \\ &= -a^2 \int_{t_{j-1}}^{t_j} \frac{\partial}{\partial \xi} G(x,\xi,t_j-\tau)\mid_{\xi=0}\; y(\tau)\; d\tau \end{aligned} \tag{18}$$

$$s_c(x,t_j) = \int_0^L G(x,\xi,t_j - t_{j-1})\; s(\xi,t_{j-1})\; d\xi \tag{19}$$

In further treatment let us replace the continuous functions $s(x,t), s_m(x,t), s_c(x,t)$ by following vectors containig the values of the functions at spatial points x_i at time instants t_j:

$$\begin{aligned} \mathbf{s}m_j^T &= [s_m(x_1,t_j),\dots,s_m(x_i,t_j),\dots,s_m(x_{nx},t_j)] \\ \mathbf{s}c_j^T &= [s_c(x_1,t_j),\dots,s_c(x_i,t_j),\dots,s_c(x_{nx},t_j)] \\ \mathbf{s}_j^T &= [s(x_1,t_j),\dots,s(x_i,t_j),\dots,s(x_{nx},t_j)] \end{aligned} \tag{20}$$

where

$$\begin{aligned} x_1 &= 0,\; x_i = x_{i-1} + \Delta x,\; i = 2,\dots,nx, \\ \Delta x &= L/(nx-1) \end{aligned}$$

In the case of the simulation model the function $s(x,t)$ represents the *true* state which was reached by the system during the time period T_m and in the case of the inversion task the function $s(x,t)$ represents the desired profile the achievement of which during the time period T_v is the goal of control.

5.1 Determination of vector $\mathbf{s}c_j$

The spatial discretization in the vector (20) is done for points $x = x_i$, i=1,2,...,nx, :

$$s_c(x_i,t_j) = \int_0^L G(x_i,\xi,T_m)\; s(\xi,t_{j-1})\; d\xi \tag{21}$$

By taking into account spline approximations of the profiles $s(x_i,t_{j-1})$:

$$s(x_i,t_{j-1}) = \sum_{k=1}^{nx} cs_k\; M_k(x_i) \tag{22}$$

where: cs_k are spline coefficients and $M_k(x_i)$ are B-splines, then it is possible to arrive to the following discrete form:

$$\begin{aligned} \mathbf{s}c_j &= \mathbf{G}_{m0}\; \mathbf{s}_{j-1} \\ \mathbf{G}_{m0} &= \mathbf{G}_{mj}\; \mathbf{M}_{n\xi}\; \mathbf{M}_{nx}^{-1} \end{aligned} \tag{23}$$

The matrix $\mathbf{G}_{m0}$ with dimensions $[nx.nx]$ can be set in advance before the control starts. For the entries of the above matrices one gets:

$$\begin{aligned} \mathbf{G}_{mj} &= \{G(x_i,\xi_k,T_m)\Delta\xi\}_{i=1,nx;\; k=1,n\xi} \\ \mathbf{M}_{nx} &= \{M_i(x_k)\}_{i=1,nx;\; k=1,nx} \\ \mathbf{M}_{n\xi} &= \{M_i(\xi_k)\}_{i=1,nx;\; k=1,n\xi} \end{aligned}$$

5.2 Determination of vector $\mathbf{s}m_j$

The elements of the vector $\mathbf{s}m_j$ are evaluated using the equation (18) written for the points $x = x_i$, i=1,2,...,nx, :

$$\begin{aligned} s_m(x_i,t_j) &= \\ &= -a^2 \int_{t_{j-1}}^{t_j} \frac{\partial}{\partial \xi} G(x_i,\xi,t_j-\tau)\mid_{\xi=0}\; y(\tau)\; d\tau \end{aligned} \tag{24}$$

Let us approximate the real measured output signal $y(t)$ (controlled boundary temperature of the bar) by spline function $v_{ym}(t)$:

$$v_{ym}(t) = \sum_{k=1}^{mt} cym_k\; M_k(t) \tag{25}$$

$$
\begin{aligned}
& v_{ym}(t) = \mathbf{m}_{mt}^T(t)\ \mathbf{c}_{ym} \\
& \mathbf{c}_{ym}^T = [cym_1, \ldots, cym_{mt}] \\
& \mathbf{m}_{mt}^T(t) = [M_1(t), \ldots, M_{mt}(t)]
\end{aligned} \tag{26}
$$

where $M_k(t)$ are suitable chosen B-splines. After substitution (26) to (24) and minor derivation the following relation is valid:

$$
\begin{aligned}
\mathbf{s}m_j &= \mathbf{G}_{mb}\ \mathbf{M}_{mt}\ \mathbf{c}_{ym} = \\
&= \mathbf{G}_{md}\ \mathbf{c}_{ym} \\
\mathbf{G}_{md} &= \mathbf{G}_{mb}\ \mathbf{M}_{mt}
\end{aligned} \tag{27}
$$

For matrix $\mathbf{G}_{mb}$ - $[nx.n\tau]$ it holds:

$$
\begin{aligned}
& \mathbf{G}_{mb} = \{gmb(i,k)\}_{i=1,nx;\ k=1,n\tau} \\
& gmb(i,k) = \\
& = -a^2 \frac{\partial}{\partial \xi} G(x_i, \xi, T_m - (k-1)\Delta\tau)\ |_{\xi=0}\ \ \Delta\tau
\end{aligned} \tag{28}
$$

the matrix $\mathbf{M}_{mt}$ - $[n\tau.mt]$ contains the values of the mention B-splines:

$$
\mathbf{M}_{mt} = \{M_k(\tau_i)\}_{i=1,n\tau;\ k=1,mt}
$$

The matrix $\mathbf{G}_{md}$ is also possible to evaluate in advance. Using the relations obtained for the vectors $\mathbf{s}c_j$ and $\mathbf{s}m_j$ we can find resulting discrete form for vector $\mathbf{s}_j$, see (17):

$$
\begin{aligned}
\mathbf{s}_j &= \mathbf{s}c_j + \mathbf{s}m_j = \\
&= \mathbf{G}_{m0}\ \mathbf{s}_{j-1} + \mathbf{G}_{md}\ \mathbf{c}_{ym}
\end{aligned} \tag{29}
$$

6 Discretization of the inversion model

The philosophy of discretization is similar with the simulation model. Now we are interested to find an optimal boundary condition - reference signal for $y(t)$ in the time interval $T_v = t_v - t_0$ with the aim to bring the state $s(x,t)$ at time instant $t_v = t_0 + T_v$ to the required state $s_z(x)$ as close as possible. Let us approximate the signal $y(t)$ by linear combination of nt B-splines with spline coefficient vector:

$$
\mathbf{c}_{yv}^T = [cyv_1, \ldots, cyv_{nt}] \tag{30}
$$

and set the required state $s_z(x)$ through a vector of its values at points x_i, i=1,2,...,nx, :

$$
\mathbf{s}z = [s_z(x_1), \ldots, s_z(x_{nx})] \tag{31}
$$

The dimensions of other vectors $(nx, n\xi, n\tau)$ are the same as in previous subsection. Similar to (29) it is possible to write:

$$
\mathbf{s}m_{ni} = \mathbf{s}z - \mathbf{s}c_{ni} \tag{32}
$$

where the vector $\mathbf{s}c_{ni}$ is calculated according to (23) for given initial state vector $\mathbf{s}_0$ using of ni steps of the simulation:

$$
\begin{aligned}
\mathbf{s}c_{ni} &= \mathbf{G}_{v0}\ \mathbf{s}_0 \\
\mathbf{G}_{v0} &= \mathbf{G}_v\ \mathbf{M}_{n\xi}\ \mathbf{M}_{nx}^{-1}
\end{aligned} \tag{33}
$$

The components of matrix $\mathbf{G}_v$ - $[nx.n\xi]$ are:

$$
\begin{aligned}
\mathbf{G}_v &= \{gv(i,j)\}_{i=1,nx;\ j=1,n\xi} \\
gv(i,j) &= G(x_i, \xi_j, T_v)\ \Delta\xi
\end{aligned} \tag{34}
$$

Based on the obtained discrete relations it is possible to approximate the smoothing functional (12) by following resulting form [4]:

$$
\begin{aligned}
& \hat{M}_\alpha[y] = \mathbf{c}_{yv}^T\ \mathbf{F}\ \mathbf{c}_{yv} - 2\ \mathbf{h}^T\ \mathbf{c}_{yv} + \\
& + \mathbf{s}m_{ni}^T\ \mathbf{s}m_{ni}\ \Delta x + \alpha\ \mathbf{c}_{yv}^T\ (\mathbf{M}_y + \mathbf{M}_{yd})\ \mathbf{c}_{yv} = \\
& = \mathbf{c}_{yv}^T\ \mathbf{R}\ \mathbf{c}_{yv} - 2\ \mathbf{h}^T\ \mathbf{c}_{yv} + \mathbf{s}m_{ni}^T\ \mathbf{s}m_{ni}\ \Delta x
\end{aligned} \tag{35}
$$

with

$$
\mathbf{R} = \mathbf{F} + \alpha\ (\mathbf{M}_y + \mathbf{M}_{yd}) \tag{36}
$$

where

$$
\begin{aligned}
\mathbf{F} &= \mathbf{G}_{vd}^T\ \mathbf{G}_{vd} \Delta x \quad \mathbf{G}_{vd} = \mathbf{G}_{vb}\ \mathbf{M}_{nt} \\
\mathbf{G}_{vb} &= \{gvb(i,k)\}_{i=1,nx;\ k=1,n\tau} \\
& \quad gvb(i,k) = \\
& \quad = -a^2 \frac{\partial}{\partial \xi} G(x_i, \xi, T_v - (k-1)\Delta\tau)\ |_{\xi=0}\ \Delta\tau \\
\mathbf{M}_{nt} &= \{M_k(\tau_i)\}_{i=1,n\tau;\ k=1,nt} \\
\mathbf{M}_y &= \Delta t \sum_{i=1}^{nt} \mathbf{m}_{\tau_i}\ \mathbf{m}_{\tau_i}^T \\
\mathbf{M}_{yd} &= \Delta t \sum_{i=1}^{nt} \mathbf{m}d_{\tau_i}\ \mathbf{m}d_{\tau_i}^T \\
\mathbf{m}_{\tau_i} &= [M_1(\tau i), \ldots, M_{nt}(\tau i)] \\
\mathbf{m}d_{\tau_i} &= [M'_1(\tau i), \ldots, M'_{nt}(\tau i)]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{s}m_{ni} &= \mathbf{G}_{vd}\ \mathbf{c}_{yv} \\
\mathbf{h}^T &= \mathbf{s}m_{ni}^T\ \mathbf{G}_{vd}\ \Delta x
\end{aligned} \tag{37}
$$

The procedure of minimization of the smoothing functional (35) can be numerically solved by various quadratic programming methods, we utilized the efficient algorithm of Powell [5]. The algorithm enables us to set various technologically inspired constraints on the vector $\mathbf{c}_{yv}$ and to limit the optimal reference signal for $y(t)$.

7 Illustrative example

The complex software for the controller and the inversion tasks was build under PC-KOS Conversational Monitor using SIC (Simulation Identification Control) library (for detail information see [2]) for the model (6) of boundary heated bar.

The heater was simulated by a second order system with time delay. The inversion tasks were solved for
- realizable profiles, it means that the required profiles $s_z(x)$ are exact solutions of equation (6)
- nonrealizable profiles, the required profiles $s_z(x)$ do not belong to the class of possible solutions of (6).

The simulations with the first type of the profiles are illustrated on Fig.1. The inversion task was solved for time instants $T_v, 2T_v$ and $3T_v$ (indicated by vertical dash lines) with $T_v = 350s$ and $T_m = 25s$. The required temperature profiles $s_z(x)$ chosen for the above time instants are marked on Fig.1c,d,e by asterisks. The corresponding optimal reference signal $y_r(t)$ obtained through stepwise solving the inversion task is drawn on Fig.1.a by a full line, the real controlled boundary temperature $y(t)$ is marked by the asterisks. The control input signal $u(t)$ to the heater is on Fig.1.b. The white noise disturbances acting on the controlled signal $y(t)$ are considered. The slight deviation between the required and reached profile in Fig.1.e is due to inconsistency of the inversion task for the time instants $2T_v$. The required profile for the time instants is fully reachable only from zero initial state.

The simulations with the nonrealizable profile are introduced on Fig.2. The results show good numerical stability of the proposed inversion algorithm for the case of inconsistent - physically not real tasks.

8 Conclusion

The procedure for regularization of the inversion task (11) is numerically tested on the problem of boundary heated bar. The optimal solution of the inversion was taken as the reference trajectory for designed predictive controller. It was found that the regularization method seek for minimum power consumption solution. The advantage of the described method was also possibility to specify the operator inaccuracy h and the zone δ for the selection of the modified state $s_m(x, t)$.

This paper describes a relatively early stage of the research in this area and gives only the main ideas which must be further elaborated for real-time implementation.

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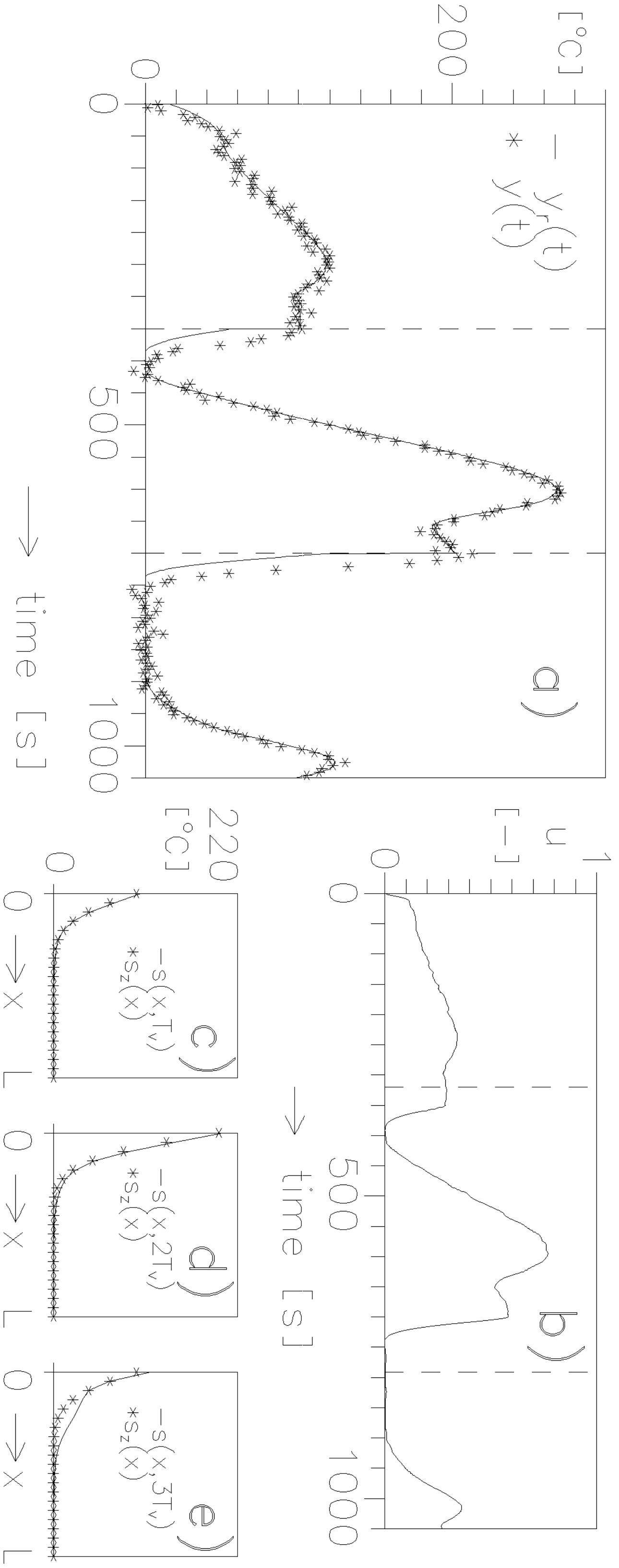

Figure 1

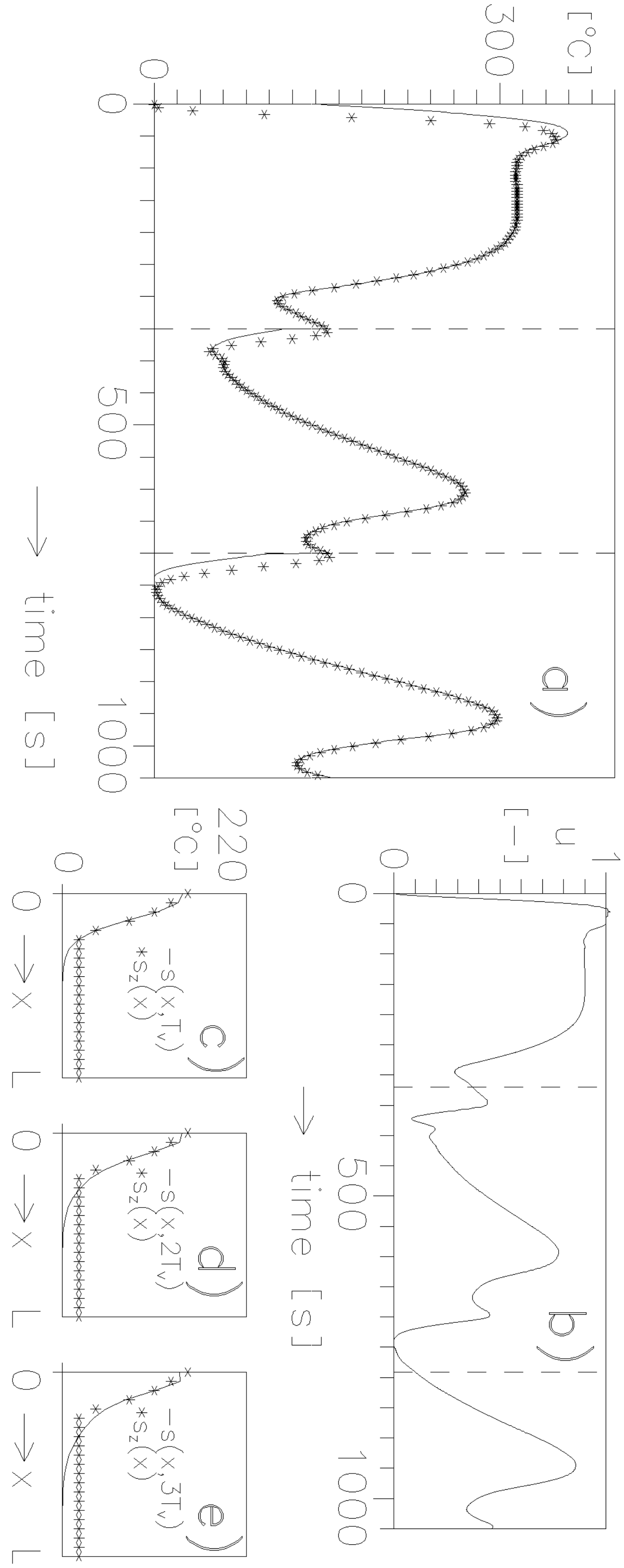

Figure 2